Sturm-Liouville Problem (part 2)

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda w(x)y = 0 \qquad a \le x \le b$$

~~So~~ subject to
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

the SL problem is regular if on $a \le x \le b$, $p(x)$, $p'(x)$, $q(x)$, $w(x)$ are continuous and $p(x) > 0$ and $w(x) > 0$

if $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all non negative, then the eigenvalues λ are also nonnegative.

if SL problem is regular, then the eigenvalues are all real and form an increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \quad \lambda_n \to \infty$ as $n \to \infty$
each is paired up with an eigenfunction y_n that are ~~both~~ mutually orthogonal with respect to the weight function $w(x)$

$$\int_a^b y_n y_m w(x)\,dx = 0 \quad \text{if } n \ne m$$

<u>example</u> $y'' + \lambda y = 0 \qquad 0 < x < L$

$y(0) = 0$

$y(L) = 0$

$$\underbrace{p = 1,\ q = 0,\ w = 1,\ \alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 1,\ \beta_2 = 0}_{\text{regular SL problem w/ nonnegative } \lambda\text{s}}$$

$\downarrow$

equivalent to $\quad X'' + \lambda X = 0$

$X(0) = X(L) = 0$

(reason for separation constant $= -\lambda$ because this is a regular SL problem)

solutions: $\quad X_n = \sin\left(\frac{n\pi}{L} x\right)$

$\lambda_n = \dfrac{n^2 \pi^2}{L^2} \qquad \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$

$n = 1, 2, 3, \ldots$

$$\int_0^L X_n X_m w(x)\,dx = \int_0^L \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{L} x\right) dx = 0$$

Fourier series is just a special case of outcome of SL problem.

example

$y'' + \lambda y = 0 \qquad 0 < x < L \qquad p=1,\ q=0,\ w=1$

$y(0) = 0 \qquad \alpha_1 = 1,\ \alpha_2 = 0$

$hy(L) - y'(L) = 0 \qquad (h>0) \qquad \beta_1 = h > 0 \quad \underbrace{\beta_2 = -1}_{\text{not nonnegative}}$

still regular SL, but since α, β are not all nonnegative, λs are not guaranteed to be nonnegative.

$\longrightarrow$ λ could be negative

we need to check if a negative λ could ~~co~~ satisfy the differential eq.

$y'' + \lambda y = 0$

assume a negative λ: let $\lambda = -k^2$ $\quad (k>0)$

k^2: avoid $\sqrt{\lambda}$ in solution (convenience only)

$y'' - k^2 y = 0$

$y = c_1 e^{kx} + c_2 e^{-kx}$

or $y = A\cosh(kx) + B\sinh(kx)$

choose whichever that's easier to work with w/ boundary conditions

$y(0) = 0 \rightarrow$ using exponentials we get $0 = C_1 + C_2$

not the best since we usually want one of those to be zero

but if we used $y = A\cosh(kx) + B\sinh(kx)$

we get $0 = A \quad \rightarrow \quad y = B\sinh(kx) \qquad B \neq 0$

use the other BC to find out more about $K \quad (\lambda = -K^2)$

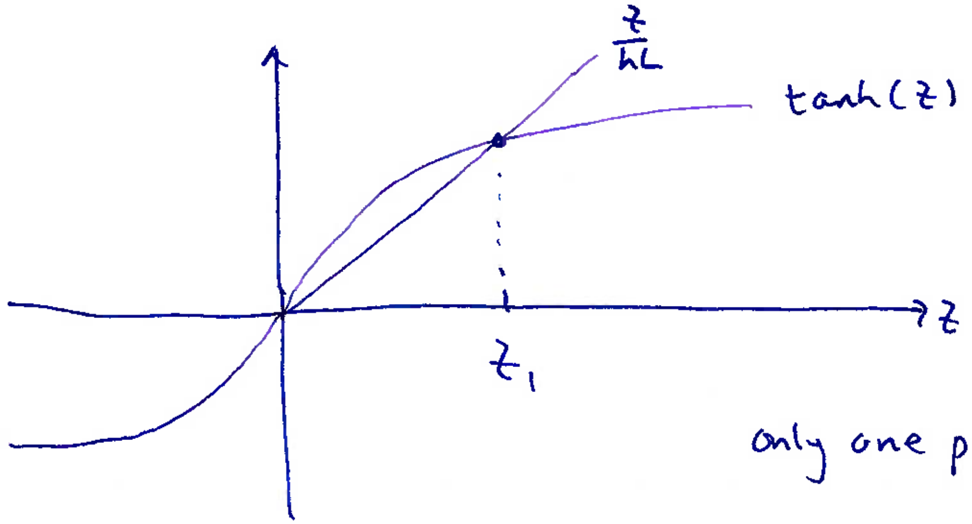
$hy(L) - y'(L) = 0 \qquad y = B\sinh(kx)$

$y' = kB\cosh(kx)$

$hB\sinh(KL) = KB\cosh(KL) \qquad B \neq 0,\ K \neq 0,\ L \neq 0$

$\tanh(KL) = \frac{K}{h}$ solve for K

$\tanh(KL) = \frac{KL}{hL} \rightarrow$ solve $\underbrace{\tanh(z) = \frac{z}{hL}}_{\text{intersections (positive) of } \tanh(z) \text{ and } \frac{z}{hL}} \qquad z = KL$



assume hL such that there is intersection

only one positive intersection : z_1

$z = kL$ and $\lambda = -k^2$

only one $z \to$ one $k \to$ one negative λ

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

negative must be ≥ 0

this negative λ is the <u>smallest</u> eigenvalue

$$\lambda_1 = -\left(\frac{z_1}{L}\right)^2$$ corresponding eigenfunction: $y_1 = \sinh\left(\frac{z_1}{L}x\right)$

next : see if $\lambda = 0$ is an eigenvalue, then $\lambda > 0$

let's revisit the heat exchange problem from last time

$$u_t = k u_{xx} \qquad 0 < x < L$$

$$u(0,t) = 0$$

$$u_x(L,t) = -h\,u(L,t) \qquad (h > 0)$$

$u(x,0) = 100$ initially heated to $u = 100$ uniformly

spatial problem : $y'' + \lambda y = 0$

$\lambda \geq 0$

$\vdots$

$\lambda_n = \dfrac{z_n^2}{L^2}$ where z_n are ~~for~~ positive intersections of $\tan(z)$ and $-\dfrac{z}{hL}$

time part solved as usual

this led to general solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n t} \sin(\sqrt{\lambda_n}\, x)$$

$u(x,0)=100$

$$100 = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x)$$

NOT sine series since $\sqrt{\lambda_n} \neq \frac{n\pi}{L}$

<u>cannot</u> use $c_n = \frac{2}{L}\int_0^L 100 \sin(\frac{n\pi}{L}x)\,dx$

but, since this is a regular SL problem, the theory guarantees

orthogonality of eigenfunctions

$$\int_0^L y_n y_m w(x)\,dx = 0 \quad \text{if } n \neq m$$

here, $w(x)=1$

$y_n = \sin(\sqrt{\lambda_n}\, x)$

find c_n:

$$100 = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x)$$

$$100 \sin(\sqrt{\lambda_m}\, x) = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x)\sin(\sqrt{\lambda_m}\, x)$$

integrate over $0 < x < L$

$$\int_0^L 100 \sin(\sqrt{\lambda_n} x)\,dx = \int_0^L C_n \left[\sin(\sqrt{\lambda_n} x)\right]^2 dx$$

all others are 0

$$\int_0^L y_n y_m \,dx = 0$$

$$\boxed{C_n = \frac{\int_0^L 100 \sin(\sqrt{\lambda_n} x)\,dx}{\int_0^L \sin^2(\sqrt{\lambda_n} x)\,dx}}$$

if $\lambda_n = \frac{n^2\pi^2}{L^2}$ then that will reduce to $\frac{2}{L}\int_0^L 100 \sin(\frac{n\pi}{L} x)\,dx$